THE COBB-DOUGLAS FUNCTION AS A FLEXIBLE FUNCTION*

A new perspective on homogeneous functions through the lens of output elasticities

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Abstract

By defining the Variable Output Elasticities Cobb-Douglas function, this article shows that a large class of production functions can be written as a Cobb-Douglas function with non-constant output elasticity. Compared to standard flexible functions such as the Translog function, this framework has several advantages. [1] It does not require the use of a second order approximation. [2] This greatly facilitates the deduction of linear input demands function without the need of involving the duality theorem. [3] It allows for a tractable generalization of the CES function to the case where the elasticity of substitution between each pair of inputs is not necessarily the same. [4] This provides a more general and more flexible framework compared to the traditional nested CES approach while facilitating the analyze of the substitution properties of nested CES functions. The case of substitutions between energy, capital and labor is provided.

Keywords: flexible production functions, Cobb-Douglas function, CES function, substitution capital-labor-energy.

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1. Introduction

In their influential contribution to economic theory, Cobb and Douglas (1928) introduced the production function that was named after them. Since, the Cobb-Douglas (CD) function has been (and is still) abundantly used by economists because it has the advantage of algebraic tractability and of providing a fairly good approximation of the production process. Its main limitation is to impose an arbitrary level for substitution possibilities between inputs. To overcome this weakness, important efforts have been made to develop more general classes of production function with as a corollary a strong increase in complexity (for a survey see e.g. Mishra, 2010).

Arrow et al. (1961) introduced the Constant Elasticity of Substitution (CES) production function which has the advantage to be a generalization of the three main functions that were used previously: the linear function (for perfect substitutes), the Leontief function (for perfect complements) and the CD function, which assume respectively an infinite, a zero and a unit elasticity of substitution (ES) between production factors.

A limitation of the CES function is known as the impossibility theorem of Uzawa (1962) - McFadden (1963) according to which the generalization of the class of function proposed by Arrow et al. (1961) to more than two factors imposes a common ES between factors. To allow for different degrees of substitutability between inputs, Sato (1967) proposed the approach of nested CES functions which has proved very successful in general equilibrium modeling and econometric studies because of its algebraic tractability. The substitution between energy and other inputs is one of the main applications (e.g. Prywes, 1986; Van der Werf, E., 2008; Dissou et al., 2015). Although this method is flexible, substitution mechanisms remain constrained and the choice of the nest structure is often arbitrary.

To overcome this limit, several "flexible" production functions have been proposed such as the Generalized Leontief (GL) (Diewert, 1971) and the Transcendental Logarithmic (Translog) function (Christensen et al., 1973)[1]. These are second order approximations of any arbitrary twice

[1] The estimation approach of a CES function using a second order approximation proposed by Kmenta (1967) is often seen as a pre-cursor to the Translog function.

differentiable production functions[2]. They have the advantage not to impose any constraint on the value of the ES between different pairs of inputs but their use is much more complex. This at least partly explains their little success in general equilibrium modeling compared to the nested CES approach[3]. Two difficulties are particularly limiting:

- Due to the complexity of flexible production functions, the demands for inputs is algebraically and computationally tedious. Using the Sheppard lemma and the duality theorem, the demands for inputs are derived from a second order approximation of the cost function at the optimum. This approach raises at least three issues. First, estimating the ES through the econometric estimation of a cost function rises important endogeneity issues since the dependent variable (the production cost) is by construction a function of the explanatory variables (the input prices). Second, all variables are generally non stationary. The risk of fallacious regression is therefore important. Third, the presence of rigidity in inputs (in particular in equipment) does not guaranty that the approximation is at the optimum. This may invalidate the key assumption underlying the Sheppard lemma and the duality theorem.
- Because of the use of a linear approximation, it is often difficult to impose the theoretical curvature conditions of the isoquants (see Diewert and Wales, 1987). This may generate poor results in the case of important variations of prices. As a consequence, the approach may be unsuitable for use in applied general equilibrium modeling because it may lead to the failure of the solver algorithm[4].

Whereas the existing literature has attempted to overcome the weakness of the CD function by proposing more general but also more complex alternatives, we remain here in the tractable framework of the CD function and investigate the conditions under which it can be used as a flexible function. We use the fact that any homogeneous production function can be written under a

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[2] For a formal proof in the case of the Translog function see e.g. Grant (1993). A theoretical discussion on this function can also be found in Thompson (2006) whereas Koetse et al. (2008) provide a meta-analysis of empirical studies estimating the substitution between capital and energy with a Translog function.

[3] See Jorgenson (1998) for the use of Translog function in general equilibrium modeling.

[4] For a discussion see Perroni and Rutherford (1995) who argue that traditional flexible functional forms suffer from an excess of flexibility. They advocate for the use of the nested CES cost function which is globally well-behaved and can provide a local approximation to any globally well-behaved cost function.

specification that is very close to a CD function. We define this specification as the Variable Output Elasticities CD function because compared to the original CD function, the output elasticities are not necessarily constant. The Output Elasticity (OE) of a given input (e.g. labor, capital or energy) measures the percentage change of output induced by the percentage change of this input. The specification of the OE is importantly influenced by the assumptions regarding the level of ES between inputs (e.g. Ferguson, 1969; Charnes et al., 1976; or Kümmel et al., 1985, 2002). The OE is constant only if the ES between every input is equal to one which is the assumption of the original CD function. For any other configuration of ES, the OE varies and depends on the relative quantities of each input and on the ES between inputs. We use here this property to characterize a relatively general class of production functions that has the advantage to combine the linear tractability of the CD function with a high level of flexibility in terms of substitution possibilities between inputs. More specifically, we shall see that this approach has the following advantages:

- It avoids the tedious algebraic of the second order approximation traditionally used in flexible functions.
- This approach allows for the derivation of algebraically tractable input demand functions without involving the duality theorem and the approximation of the cost function at the optimum.
- This greatly facilitates the deduction of linear input demands that can be estimated using standard linear regression models.
- This new class of function allows for a generalization of the CES to the case where the ES between each pair of inputs are not necessarily the same and hence for avoiding the limitation of the impossibility theorem and the use of the nested CES approach. This may prove very useful to analyze the substitution phenomena between energy and other inputs.
- This allows for easily introducing different levels of ES between production factors. In particular, changing the level of elasticity between factors is easier than in the nested CES approach since it does not require changing the structure of the nest. Moreover, relevant constrains on the ES parameters allow for reproducing the particular case of a nested CES function.

Section 2 defines formally the Variable Output Elasticities CD (VOE-CD) function in the general case of *n* inputs and shows that the CD function can be seen as a flexible function generalizing any homogeneous function. Section 3 shows that the VOE-CD provides a generalization of the CES function where the ES between each pair of inputs are not necessarily the same. Section 4 derives the demand for inputs that minimizes the production costs in the case of a VOE-CD production function. Section 5 investigates the particular case of a nested CES function with 3 inputs (e.g. capital-labor-energy) and shows that its VOE-CD formulation allows for a straightforward analysis of the substitution properties of a system of nested CES functions. Section 6 concludes.

2. The Variable Output Elasticities Cobb-Douglas function

In order to characterize the VOE-CD function, let us first define the general characteristic of the technology of production (Definition 1):

DEFINITION 1. The production function is:

1. a continuous and twice differentiable function Q[5]:

$$Q = Q(X_1, X_2, ..., X_i, ..., X_n) \quad (1)$$

Where X_i is the quantity of input (or production factor) $i \in [1;2;...;n]$ used to produce the quantity of production (or output) Q.

2. homogeneous of degree 1 (constant returns-to-scale)[6]

3. increasing in inputs: $Q'(X_i) = \frac{\partial Q}{\partial X_i} > 0$

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[5] In this paper, the first and second partial derivatives of the function Q with respect to X_i are respectively $Q'(X_i) = \frac{\partial Q}{\partial X_i}$ and $Q''(X_i) = \frac{\partial^2 Q}{(\partial X_i)^2}$. Variables in growth rate are referred to as $\dot{X} = \frac{\mathrm{d}X}{X} = \frac{\mathrm{d}(\ln X)}{\mathrm{d}X}$. All parameters written in Greek letter are positive.

[6] If one assumes that X_i is the quantity of "efficient" input and $Q = Y^{1/\theta}$ where Y is the level of production and θ the level of returns-to-scale, all the results presented below can be generalized to account for increasing/decreasing returns-to-scale and technical progress.

4. strictly concave (reflecting the law of diminishing marginal returns): $Q''(X_i) = \frac{\partial^2 Q}{(\partial X_i)^2} < 0$ and $\frac{\partial\left(Q'(X_i)\right)}{\partial X_j} = \frac{\partial^2 Q}{\partial X_i \partial X_j} > 0$ for $i \neq j$ where $i, j \in [1;2;...;n]$.

PROPOSITION 1. Any homogeneous production function as defined in Definition 1 can be written as follows:

$$\dot{Q} = \sum_{i=1}^{n} \varphi_i \dot{X}_i \Leftrightarrow \mathrm{d}(\ln Q) = \sum_{i=1}^{n} \varphi_i \, \mathrm{d}(\ln X_i) \tag{2}$$

with

$$\varphi_i = \frac{Q'(X_i)X_i}{\sum_{j=1}^{n} Q'(X_j)X_j} = \left[\sum_{j=1}^{n}\left(\frac{Q'(X_j)X_j}{Q'(X_i)X_i}\right)\right]^{-1} \tag{3}$$

Where $\varphi_i \in [0;1]$ is the output elasticity (OE) of input *i*. It measures the relative change in output induced by a relative change in input *i*. Moreover, $\sum_{i=1}^{n} \varphi_i = 1$ [because of Equation (3)] [7].

PROOF: The total differential of the production function (1),

$$\mathrm{d}\,Q = \sum_{i=1}^{n} \frac{\partial Q}{\partial X_i} \mathrm{d}\,X_i \tag{4}$$

can be rewritten in growth rate:

$$\frac{\mathrm{d}Q}{Q} = \sum_{i=1}^{n} \frac{Q'(X_i)X_i}{Q} \frac{\mathrm{d}X_i}{X_i} \Leftrightarrow \dot{Q} = \sum_{i=1}^{n} \frac{Q'(X_i)X_i}{Q} \dot{X}_i \tag{5}$$

The Euler's Theorem states that a function which is homogeneous of degree 1 can be expressed as the sum of its arguments weighted by their first partial derivatives:

[7] This comes from Definition 1.2, that is from the hypothesis of constant returns-to-scale.

$$Q = \sum_{j=1}^{n} Q'(X_j)X_j \tag{6}$$

Incorporating (6) into (5), we see that (4) can equivalently be written as Equations (2) and (3). □

The specification of Equation (2) is very close to the one of a CD function. In both cases, the specification is log-linear and the OEs, φ_i, appear explicitly. There are however two major differences: (a) the CD function is formulated in level whereas Equation (2) is written in logarithmic first difference (which is equivalent to a growth rate); (b) in the standard CD function written in level, the OEs are constant whereas they are not necessarily constant in Equation (2). For this reason and in order to avoid any ambiguity, we shall from now on adopt the following definition.

DEFINITION 2. The VOE-CD function and the COE-CD function

1. A production function is a **Variable** Output Elasticities Cobb-Douglas (**V**OE-CD) function if it is specified as Equations (2) and (3).

2. A production function is a **Constant** Output Elasticities Cobb-Douglas (**C**OE-CD) function if it is specified as:

$$Q = \prod_{i=1}^{n} X_i^{\varphi_i} \Leftrightarrow \ln Q = \sum_{i=1}^{n} \varphi_i \ln X_i \tag{7}$$

Where the OEs, φ_i, are constant.

3. A production function is a VOE-CD function **in level** if it is specified as Equations (7) and (3).

PROPOSITION 2.

1. A VOE-CD function as defined in Definition 2.1 is equivalent to a COE-CD function as defined in Definition 2.2 if the OEs, φ_i, are all constant.

2. A VOE-CD function in level as defined in Definition 2.3 is **not** equivalent to a VOE-CD function as defined in Definition 2.1. The higher the changes in the OEs, φ_i, are, the higher the gap between the VOE-CD function (Definition 2.1) and the VOE-CD function in level (Definition 2.3).

PROOF:

1. Taking the integral of Equation (2) assuming that φ_j are constant for all $i=[1;n]$ leads to the specification in level (7): $\int \mathrm{d}(\ln Q) = \ln Q \equiv \sum_{i=1}^{n} \varphi_i \int \mathrm{d}(\ln X_i) = \sum_{i=1}^{n} \varphi_i \ln X_i$ (the constants of integration are set to zero for algebraic simplicity).

2. Taking the total derivative of Equation (7) leads to,

$\mathrm{d}(\ln Q) = \sum_{i=1}^{n} \left(\frac{\partial \ln Q}{\partial \ln X_i} \mathrm{d}(\ln X_i) + \frac{\partial \ln Q}{\partial \varphi_i} \mathrm{d}(\varphi_i) \right) = \sum_{i=1}^{n} \left(\varphi_i \mathrm{d}(\ln X_i) + (\ln X_i).\mathrm{d}(\varphi_i) \right)$ which is not equivalent to Equation (2). The VOE-CD function in level (Definition 2.3) tends toward a VOE-CD function (Definition 2.1) if $\mathrm{d}(\varphi_i) \to 0$. □

Proposition 2.1 shows that the VOE-CD function encompasses the case of the COE-CD function. This simply comes from the fact that the COE-CD function is one particular type of degree 1 homogeneous function whereas the VOE-CD function can serve as a generalization of all type of degree 1 homogeneous function. Proposition 2.2 shows that the VOE-CD function in level, Equation (7), leads to a different outcome than the VOE-CD function (2) even if one allows for the OEs to vary. It can therefore not be used as a generalization of all type of degree 1 homogeneous function. The VOE-CD function in level (7) provides a poor approximation of the specification in logarithmic first difference (2) in the case of an important change in the ratio between marginal productivities (i.e. between input prices).

Proposition 1 can more or less explicitly be found in the literature. In particular, several authors have proposed to reformulate a degree 1 homogeneous production function [as Equation (1)] in growth rate [as Equation (2)]. Ferguson (1969, pp. 76-83) uses this reformulation to analyze the properties of homogeneous production functions. He uses the concept of OE to construct what he calls the "function coefficient" which is defined as the elasticity of output with respect to a proportional changes in all inputs. He shows that the function coefficient is greater (resp. equal, smaller) than/to one in the case of increasing (resp. constant, decreasing) returns to scale. He also derives explicitly the specification of the OE in the case of a CES function. Kümmel et al. (1985, 2002) derive the explicit specification of the OE in the case of a Translog and a LINEX function with three inputs. For both cases, they show that the OE of each input is a function of the input ratios between labor, capital and energy which is a major difference compared to the CD function where every OE is constant.

Charnes et al. (1976) do not derive Equation (2) but show that any degree 1 homogeneous production function can be reformulated as a form close to a CD function: $Q = H.\prod_{i=1}^{n} X_i^{\varphi_i}$ where $H = \prod_{i=1}^{n} \left(\frac{Q'(X_i)}{\varphi_i} \right)^{\varphi_i}$ and the OEs, φ_i, are defined as in Equation (3). To do so, they simply rewrite the level of production as follows: $Q = \prod_{i=1}^{n} Q^{\varphi_i} \left(\frac{X_i}{X_i} \right)^{\varphi_i} = \prod_{i=1}^{n} \left(\frac{Q}{X_i} \right)^{\varphi_i} X_i^{\varphi_i}$ (using the fact that $\sum_{i=1}^{n} \varphi_i = 1$). They use also Equation (3) and therefore the Euler's Theorem to reformulate the ratio $\frac{Q}{X_i} = \frac{Q'(X_i)}{\varphi_i}$. Although in appearance simple, this proposed CD formulation in level is in fact complicated because of the complexity of *H*. This limits its use for concrete applications and may explain why the authors do not draw further results except using directly Equation (3) to derive the specification of the OEs in the case of CD and CES functions with 2 inputs (in the extended version of their paper).

On the contrary, the present study derives additional practical implications from Proposition 1 compared to the existing literature. To do so, it uses the fact that the OE can be formulated as a function of (the sum of) the ratio between the marginal productivities of each pair of inputs times the ratio between the same pair of inputs: $\frac{Q'(X_j)X_j}{Q'(X_i)X_i}$ (see the right hand side of Equations (3)). The papers quoted above do not proceed to this reformulation. For Ferguson (1969) and Kümmel et al. (1985, 2002), the reason is that they do not use the Euler's Theorem to reformulate Q in the denominator of the OE specification in Equation (5) which is only the intermediary step of the proof that leads to Equations (3) of Proposition 1. Charnes et al. (1976) do use the Euler's Theorem but do not proceed to the reformulation which as we shall see is very useful for two reasons: (1) the notion of ES imposes a link between the input ratio and their marginal productivity; (2) profit maximization implies that at the optimum, the ratio between the marginal productivities of two inputs equals the ratio between their prices.

One can therefore expect to draw additional results compared to the existing literature from Proposition 1. We shall indeed see that this allows for deriving a generalization of the CES function where the ES between each pair of inputs are not necessarily equal (Proposition 3). This provides a new perspective regarding homogeneous functions and the concept of OE, allowing for the derivation of a flexible function that is algebraically more tractable than existing flexible production function. In particular, the resulting demand for inputs is linear and more straightforward for the analysis of substitution mechanisms between inputs compared to existing flexible function (see section 4 and 5).

3. The VOE-CD function as a generalization of the CES function

To the best of our knowledge, the current paper is the first to derive these additional implications of the Euler's Theorem which may prove very promising both for the theoretical and empirical analysis of production functions. To understand the underlying intuition, recall that the specification presented in Proposition 1 (i.e. Equations (2) and (3)) is perfectly equivalent to the total differential of the production function (1) (i.e. Equation (4)) as shown in the proof. Assuming that production and the other inputs are constant ($\mathrm{d}Q = \mathrm{d}X_{\mathrm{k}} = 0$ for $\mathrm{k} \neq i, j$), Equation (4) can be reformulated into the textbook specification of the Marginal Rate of Substitution (MRS) between inputs j and i (e.g. Varian, 1992), where $MRS_{ji} = \dfrac{\mathrm{d}X_j}{\mathrm{d}X_i}$. This reformulation of Equation (4) shows that the MRS between inputs j and i is equal to the ratio between their marginal productivity:

$$MRS_{ji} = \frac{\mathrm{d}X_j}{\mathrm{d}X_i} = -\frac{Q'(X_i)}{Q'(X_j)} \tag{8}$$

The MRS being the first derivative of the isoquant (the slope of the iso-production curve), its integral is the isoquant itself. We can use this property to derive various classes of production functions by formulating hypothesis about the specification of the marginal productivity of each input. For instance, in the case of perfect substitutes, the MRS is constant and the isoquant is a straight line. For less substitutable input, the MRS is increasing and the isoquant is more convex. Assuming a single reference point where the combination for the levels of production and inputs is known, the integral of the MRS from this point allows for drawing any isoquant and thus for deriving any production function. Because of the strict equivalence between Proposition 1 and Equation (4), we shall see that this amounts to formulating hypothesis regarding the specification of the OEs. To show this more formally, let us introduce the definition of the ES proposed by Hicks (1932) and Robinson (1933).

DEFINITION 3. The ES of Hicks (1932) and Robinson (1933) between inputs *i* and *j* (η_{ij}) measures the change in the ratio between two factors of production due to a change in their relative marginal productivity, i.e. in the MRS. Its specification is:

$$\begin{aligned} -\eta_{ij} = \frac{\mathrm{d}\ln(X_i / X_j)}{\mathrm{d}\ln\left(Q'(X_i)/Q'(X_j)\right)} \Leftrightarrow \frac{Q'(X_i)}{Q'(X_j)} = \frac{\tilde{\varphi}_i}{\tilde{\varphi}_j}\left(\frac{X_i}{X_j}\right)^{-1/\eta_{ij}} \\ \Leftrightarrow \dot{X}_i - \dot{X}_j = -\eta_{ij}\left(\dot{Q}'(X_i) - \dot{Q}'(X_j)\right) \end{aligned} \quad (9)$$

Where $\tilde{\varphi}_i$ is a constant representing the relative weight of input *i* in the production function: $\sum_{i=1}^{n}\tilde{\varphi}_i = 1$[8]. For algebraic convenience, the constant resulting from the integration in the second equality in Equation (9) is defined as the ratio $\frac{\tilde{\varphi}_i}{\tilde{\varphi}_j}$[9].

[8] In applied general equilibrium models, these weights are calibrated at a reference point in time using base year data.

[9] The second equality in Equation (9) is derived from: $\int \mathrm{d}\ln\left(\frac{Q'(X_i)}{Q'(X_j)}\right) \equiv -\frac{1}{\eta_{ij}}\int \mathrm{d}\ln\left(\frac{X_i}{X_j}\right)$. This leads to $\ln\left(\frac{Q'(X_i)}{Q'(X_j)}\right) + C_0 = \ln\left(\frac{X_i}{X_j}\right)^{-\frac{1}{\eta_{ij}}} + C_1 \Leftrightarrow \frac{Q'(X_i)}{Q'(X_j)} = \exp(C_1 - C_0)\left(\frac{X_i}{X_j}\right)^{-\frac{1}{\eta_{ij}}}$, where C_0 and C_1 are two constants of integration.

To be economically meaningful, one often expects the sign of the ES defined in Definition 3 to be negative ($-\eta_{ij} < 0$)[10]: a 1% increase in the ratio between the marginal productivities (or the prices) of two inputs leads to a η_{ij}% decrease in the ratio between these inputs because the producer has the incentive to substitute toward the input that became relatively cheaper. The parameter η_{ij} is therefore expected to be positive. Unless stated otherwise, for convenience the term ES will refer from now on to the parameter η_{ij} that is to the negative of the ES (or its absolute value). Notice also that this definition of the ES is symmetric: $\eta_{ji} = \eta_{ij}$[11].

PROPOSITION 3. The combination of the definition of the ES (Definition 3) to the definition of the VOE-CD function (Definition 2.1) provides a generalization of the CES function where the ES between each pair of inputs are not necessarily the same and where the OE of input *j* is:

$$\varphi_i = \left[\sum_{j=1}^{n} \left(\frac{\tilde{\varphi}_j}{\tilde{\varphi}_i} \left(\frac{X_j}{X_i} \right)^{1-1/\eta_{ij}} \right) \right]^{-1} \tag{10}$$

PROOF:

Integrating (9) into (3) by solving for the ratios between the marginal productivities leads to Equation (10) □

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[10] We shall see in Section 5 that in the case of more than two inputs, there is a meaningful economic reason for the ES to have a positive sign.

[11] Extrapolating the results presented below to the asymmetric case using the definition of the ES proposed by Morishima and advocated by Blackorby and Russell (1989) is straightforward but complicates their algebraic exposition. This generalization requires to change Equation (9) into $\dot{X}_i - \dot{X}_j = -\eta_{ij}\dot{Q}'(X_i) + \eta_{ji}\dot{Q}'(X_j)$ with $\eta_{ji} \neq \eta_{ij}$.

COROLLARY 1. Under the assumption of a CES function, the ES is common between each pair of input: $\eta_{ij} = \eta$ for all i, j.

1. The specification of the OE is:

$$\varphi_i = \left[\sum_{j=1}^{n} \left(\frac{\tilde{\varphi}_j}{\tilde{\varphi}_i} \left(\frac{X_j}{X_i} \right)^{1-1/\eta} \right) \right]^{-1} \tag{11}$$

2. If the ES is equal to one ($\eta = 1$), the OEs are constant: $\varphi_i = \left(\sum_{j=1}^{n} \frac{\tilde{\varphi}_j}{\tilde{\varphi}_i} \right)^{-1} = \frac{\tilde{\varphi}_i}{\sum_{j=1}^{n} \tilde{\varphi}_j}$. The VOE-CD collapses into a COE-CD function (Definition 2.2).

3. If the ES tends to zero ($\eta \to 0$), the OE can take the following values: $\varphi_i = \frac{\tilde{\varphi}_i}{\sum_{j=1}^{n} \tilde{\varphi}_j}$ if $\frac{X_i}{X_j} = 1$ whereas $\varphi_i \to 0$ (resp. 1) if $\frac{X_i}{X_j} > 1$ (resp. <1). The VOE-CD function tends toward a Leontief function that characterizes perfect complements.

4. If the ES tends toward infinity ($\eta \to +\infty$), the OE tends toward $\varphi_i = \frac{\tilde{\varphi}_i X_i}{\sum_{j=1}^{n} \tilde{\varphi}_j X_j}$. The VOE-CD function tends toward the linear production function that characterizes perfect substitutes: $\mathrm{d}Q = \sum_{i=1}^{n} \tilde{\varphi}_i \,\mathrm{d}X_i \Leftrightarrow Q = \sum_{i=1}^{n} \tilde{\varphi}_i X_i$.

PROOF:

1. Straightforward from Equation (10). 2. to 4. Straightforward from Equation (11). □

Corollary 1.2 can be explained by the fact that with an ES equal to one, any change in the ratio between two inputs is exactly compensated by the change in their relative marginal productivity (see Equation (9)), so that the OE is always constant. Corollary 1.3 reflects the perfect complementary between inputs: increasing the quantity of input *i* while leaving the quantity of the other inputs constant does not increase the level of production because the marginal productivity of input *i* falls to zero (thus $\varphi_i \to 0$); increasing the quantity of the other inputs *j* while leaving the quantity of the input *i* constant does not increase the level of production either but increases the marginal productivity of input *i* (thus $\varphi_i \to 1$); the OEs stay constant only if the quantities of every input increase in the same proportion. Corollary 1.4 reflects the perfect substitutability between inputs where the marginal productivity of each input is always constant and equal to $\tilde{\varphi}_i$ whatever the level of the ratio between inputs is.

4. The demand for inputs

We now deduce the demand for inputs in the case of the VOE-CD production function (2) and (3). Driven by a maximizing profit behavior, the producer chooses her demand for each input by minimizing her production cost (12) subject to the technical constraint (1):

$$C = \sum_{i=1}^{n} P_i^X X_i \tag{12}$$

Where P_i^X is the price of input *j*. The Lagrangian to this problem is:

$$L = C - \lambda\left(Q - Q(X_i)\right) \tag{13}$$

The well-known first order necessary conditions ($L'(X_i) = 0$) say that at the optimum, the ratio between the marginal productivities of two inputs equals the ratio between their prices[12]:

$$Q'(X_i) / Q'(X_j) = P_i^X / P_j^X \tag{14}$$

The combination of Equations (3) and (14) shows that, at the optimum, the OE of Input *i* in the VOE-CD function corresponds to the cost share of input *i*:

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[12] The first order conditions are sufficient for optimality because of the assumption of a strictly concave production function (Definition 1.4).

$$\varphi_i = \frac{P_i^X X_i}{\sum_{j=1}^{n} P_j^X X_j} \tag{15}$$

Under the assumption that the sales' revenues of production are totally exhausted by the remuneration of the factors of production, Equation (15) is also the share (in value) of input *i* in the production and allows for calibrating the VOE-CD function (2) at a base year in the exact same way it is customary to calibrate a COE-CD function[13].

Combining the first order conditions (14) to the definition of the ES (9) and the production function (2) gives the demand for each factor as a positive function of output and a negative function of the relative prices between production factors (see Appendix A)[14]:

$$\dot{X}_i = \dot{Q} - \sum_{\substack{j=1 \\ j \neq i}}^{n} \eta_{ij} \varphi_j (\dot{P}_i^X - \dot{P}_j^X) \tag{16}$$

Assuming a constant ES between inputs, $\eta_{ij} = \eta$ for all *i* and *j*, the demand for production factors (16) expectedly simplifies to the specification that is derived from a CES function. The input demand depends only on the relative price between the input price and the average input price index, P^Q (which corresponds also to the production price under the assumption of profit exhaustion):

$$\dot{X}_i = \dot{Q} - \eta(\dot{P}_i^X - \dot{P}^Q) \tag{17}$$

with $$\dot{P}^Q = \sum_{i=1}^{n} \varphi_i \dot{P}_i^X \tag{18}$$

[13] This standard calibration procedure is partly at the origin of the controversy about the robustness of the empirical success of the COE-CD function. According to Samuelson (1979), the CD econometric estimation would do nothing more than reproducing the income distribution identity (for a literature review see e.g. Felipe and Adams, 2005). This controversial issue is largely beyond the scope of the current paper. We keep it for future research.

[14] Because of the impossibility theorem, a common ES between all factors is the only possible case where the ES are constant between every pair of inputs. When the ES differs between pairs, at least one ES is not constant. The reason is that the system of Equations (9) is over-identified for a number of inputs higher than 2 ($n > 2$).

One may notice that the above specification is similar to the consumer's demand for goods derived from a CES utility function. Here the price index (P^Q) is nothing else but the linear formulation of the Dixit and Stiglitz (1977) CES price index (e.g. Blanchard and Kiyotaki, 1987).

As we did in Lemoine et al. (2010) in the case of two inputs (labor and capital), the factor demand Equations (16) and (17) can also be obtained by linearizing the input demands derived from a CES production function. Using quarterly data over the 1970-2007 period for the euro area, this study also estimates that the ES between capital and labor is between 0.3 and 0.4 depending on the specification of the relative cost between labor and capital. This result hardly changes whereas we assume that the OEs are constant or that they vary according to the shares of value added going to labor and capital. The reason is that these shares are empirically relatively stable over time compared to the fluctuations of the relative cost between labor and capital.

5. Substitution properties of a nested CES system: the case of 3 inputs (capital-labor-energy)

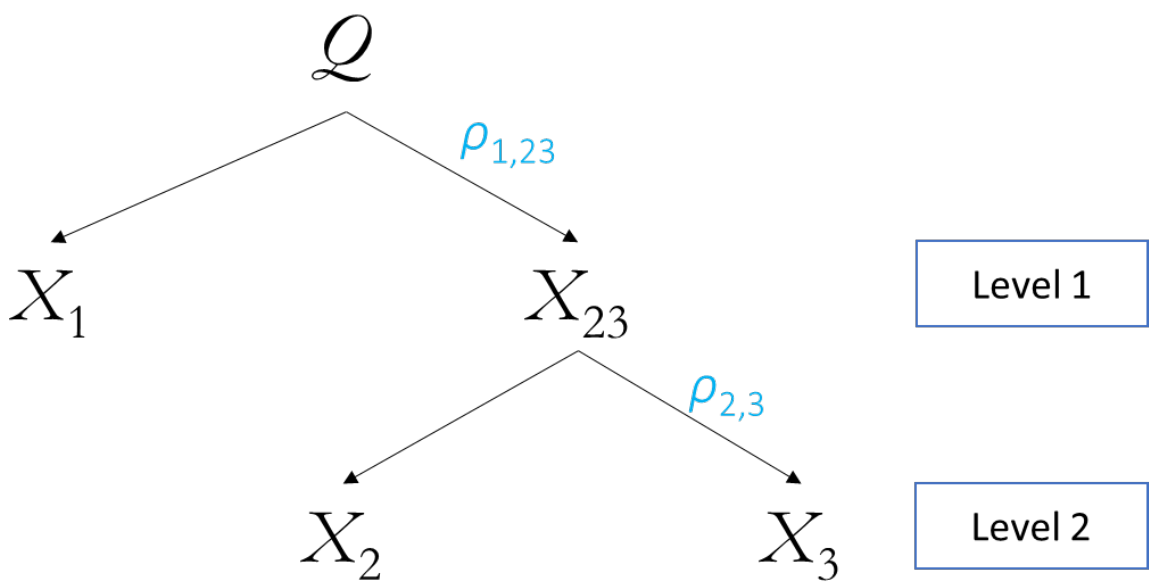
The input demand derived from a VOE-CD function (Equation (16)) can be used to represent a nested production function structure and has two advantages compared to a system based on nested CES functions. First, it is more general since the CES function is a particular case of the VOE-CD function. Second, it is more tractable because the input demand derived from a VOE-CD function is linear. This has the advantage to allow for a straightforward analysis of the substitution properties of a system of nested functions.

As an illustration, we shall now reproduce a nested CES structure with a VOE-CD nested structure. Figure 1 shows a textbook case of a two-level nested CES functions with 3 inputs abundantly used in general equilibrium modeling and econometric studies. These inputs generally refer to labor (X_1), capital (X_2) and energy (X_3): also known as KLE (e.g. Prywes, 1986; Van der Werf, 2008)[15]. With these three inputs, several combinations of nested structure are possible. The choice is rather arbitrary but it has often important implication on the substitution properties of the

[15] The more tedious case of an example of nested structure with 4 inputs is derived in Appendix B.

model (for a discussion see Van der Werf, 2008). For the purpose of our illustration, this choice has no consequences on the conclusions presented below. Let us assume that at the first level, labor (X_1) can be substituted with the aggregate capital/energy (X_{23}) with an ES of $\rho_{1,23}$. At the second level, capital (X_2) can be substituted to energy (X_3) with an ES of $\rho_{2,3}$.

Figure 1. Example of a nested CES function with 3 inputs



The demand for input (16) can be used to represent this nested structure, replacing eventually Q and P_j^X by the relevant aggregate. This leads to the following linear system of equations:

$$\dot{X}_1 = \dot{Q} - \rho_{1,23}\varphi_{23/123}(\dot{P}_1^X - \dot{P}_{23}^X) \quad (19)$$

$$\dot{X}_2 = \dot{X}_{23} - \rho_{2,3}\varphi_{3/23}(\dot{P}_2^X - \dot{P}_3^X) \quad (20)$$

$$\dot{X}_3 = \dot{X}_{23} - \rho_{2,3}\varphi_{2/23}(\dot{P}_3^X - \dot{P}_2^X) \quad (21)$$

$$\dot{X}_{23} = \dot{Q} - \rho_{1,23}\varphi_{1/123}(\dot{P}_{23}^X - \dot{P}_1^X) \quad (22)$$

Where $\varphi_{23/123} = (1-\varphi_1)$ is the share of the aggregate X_{23} into the production and $\varphi_{3/23} = \varphi_3 / (1-\varphi_1)$ is the share of the input X_3 into the aggregated X_{23}. Following the same logic, $\varphi_{2/23} = \varphi_2 / (1-\varphi_1)$, $\varphi_{1/123} = \varphi_1$. $\dot{P}_{23}^X = (\varphi_2 \dot{P}_2^X + \varphi_3 \dot{P}_3^X) / (1-\varphi_1)$ is the price of the aggregate X_{23}.

By integrating Equation (22) into (21) and (20), it is straightforward to derive the explicit production factors demand as defined in (16) with $n = 3$:

$$\begin{aligned} \dot{X}_1 &= \dot{Q} - \eta_{1,2}\varphi_2(\dot{P}_1^X - \dot{P}_2^X) - \eta_{1,3}\varphi_3(\dot{P}_1^X - \dot{P}_3^X) \\ \dot{X}_2 &= \dot{Q} - \eta_{1,2}\varphi_1(\dot{P}_2^X - \dot{P}_1^X) - \eta_{2,3}\varphi_3(\dot{P}_2^X - \dot{P}_3^X) \\ \dot{X}_3 &= \dot{Q} - \eta_{1,3}\varphi_1(\dot{P}_3^X - \dot{P}_1^X) - \eta_{2,3}\varphi_2(\dot{P}_3^X - \dot{P}_2^X) \end{aligned} \tag{23}$$

We find that the ES between each pair of inputs implicitly defined by the nested system (19)-(22) is:

$$\begin{aligned} \eta_{1,2} &= \eta_{1,3} = \rho_{1,23} \\ \eta_{2,3} &= \frac{\rho_{2,3} - \rho_{1,23}.\varphi_1}{1-\varphi_1} \end{aligned} \tag{24}$$

This result allows for analyzing straightforwardly the substitution properties of the nested system when the relative price between input changes. Because of the strong non linearity of the CES function, such an analysis using directly the nested CES system may prove very cumbersome. Because Input 2 and 3 are part of the same aggregate at the first level of the nest, the ES between Input 1 and the other inputs are all equal (see Equation (24)). A decrease of the price of Input 1 leads to an unambiguous increase of its demand to the detriment of the other inputs. Because they are defined at the second level in the nest, the sign of the ES between Input 2 and 3 is ambiguous. The sign of $\eta_{2,3}$ may be negative: an increase of the price of Input 2 relatively to the price of Input 3 may therefore lead to a decrease of the demand for Inputs 3 whereas Input 2 and 3 are substitutes in level 2. This seemingly unintuitive result comes from the first level of the nest, where the aggregate 23 is a substitute to Input 1. Increasing the price of input 2 leads to a higher price of the aggregate 23 and therefore to substitutions of Inputs 2 and 3 to Input 1. Depending on the share of input 1 into the production (φ_1) and on the level of ES in the first and second nest ($\rho_{1,23}, \rho_{2,3}$), the quantity of Input 3 may decrease. Equation (24) shows that this unintuitive effect is avoided if $\rho_{2,3} > \rho_{1,23}.\varphi_1$.

Therefore choosing a higher level of ES at the second level of the nest ($\rho_{2,3} > \rho_{1,23}$) will ensure that the increase of the price of one input always leads to a decrease in its demand. But if Input 2 and 3 are complementary inputs, that is if $\rho_{2,3}$ is close to zero, this unintuitive result is likely to arise. Therefore complementarity between production factors are often mentioned in the literature to justify negative ES. The most famous example is the complementarity between capital and energy (see e.g. Berndt and Wood, 1979; Frondel and Schmidt, 2002; Roy et al., 2006).

6. Conclusions

This article has defined the VOE-CD function and shown that this function can be used to formulate any homogeneous production function. This framework appears to have several advantages. First, it is relatively simple compared to most alternative approaches while allowing a wide range of substitution possibilities. It provides a linear formulation and thus avoids the tedious algebraic of the second order approximation used in flexible functions such as the Translog function. It allows for the derivation of linear input demand functions without involving the duality theorem which holds only at the optimum. Second, it provides a generalization of the CES function to the case where the ES between each pair of inputs are not equal. Third, its tractability allows for a straightforward analysis of the substitution properties of a system of nested functions.

Moreover, this approach has potentially several very useful applications. As it leads to linear input demands that are general in terms of substitution possibilities, it may prove promising in the econometric analysis of the producer. In this respect, the attempt made by Lemoine et al. (2010) to estimate a VOE-CD function in the euro area in the case of two inputs (labor and capital) gave promising results. It could be extended to account for energy substitutions and by applying the approach developed by León-Ledesma et al. (2010) that allows for a robust and joint identification of the ES and the biased technical change parameters.

Applied general equilibrium models provide another important application. As in the multi-sector macroeconomic model ThreeME (Callonnec et al., 2013; Landa et al., 2016; Bulavskaya and Reynès, 2018), the input demands derived from a VOE-CD function can easily be introduced to model the substitutions between energy, capital, labor and material but also between energy sources (electricity, petrol, etc.). Compared to the nested CES approach, it allows for testing alternative substitution hypotheses without changing the nest structure of the model. Compared to the use of the traditional flexible functions (such as the Translog function), it has the advantage to provide tractable and well-behaved input demands.

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References

Arrow, K. J., Chenery, H. B., Minhas, B. S., Solow, R. M., 1961. Capital-Labor Substitution and Economic-Efficiency. Review of Economics and Statistics 43 (3), 225-250.

Berndt E. R., Wood D. O., 1979. Engineering and Econometric Interpretations of Energy-Capital Complementarity. American Economic Review, 69 (3), 342-354.

Blackorby, C., Russell, R. R., 1989. Will the Real Elasticity of Substitution Please Stand Up? (A Comparison of the Allen/Uzawa and Morishima Elasticities). American Economic Review 79 (4), 882-888.

Blanchard, O. J., Kiyotaki, N., 1987. Monopolistic Competition and the Effects of Aggregate Demand. The American Economic Review 77 (4), 647-666.

Bulavskaya, T., Reynès, F., 2018, Job creation and economic impact of renewable energy in the Netherlands, Renewable Energy 119, 528-538.

Callonnec, G., Landa, G., Malliet, P., Reynès, F., Tamsamani, Y. Y., 2013. A Full Description of the THREEME Model: Multi-sector Macroeconomic Model for the Evaluation of Environmental and Energy policy. Retrieved from www.ofce.sciences-po.fr/indic&prev/modele.htm

Charnes, A., Cooper, W. W., Schinnar, A. P., 1976. A theorem on homogeneous functions and extended Cobb-Douglas forms. Proc. Natl. Acad. Sci. 73 (10), 3747–3748.

Christensen, L. R., Jorgenson, D. W., Lau, L. J., 1973. Transcendental Logarithmic Production Frontiers. The Review of Economics and Statistics 55 (1), 28-45.

Cobb, C. W., Douglas, P. H., 1928. A Theory of Production. The American Economic Review 18 (1), 139-165.

Diewert, W. E., 1971. An Application of the Shephard Duality Theorem: A Generalized Leontief Production Function. The Journal of Political Economy 79 (3), 481-507.

Diewert, W. E., Wales, T. J., 1987. Flexible Functional Forms and Global Curvature Conditions. Econometrica 55 (1), 43-68.

Dissou, Y., Karnizova, L., Sun, Q., 2015. Industry-level Econometric Estimates of Energy-Capital-Labor Substitution with a Nested CES Production Function. Atlantic Economic Journal 43(1),107-121.
Dixit, A. K., Stiglitz, J. E., 1977. Monopolistic Competition and Optimum Product Diversity. The American Economic Review 67 (3), 297-308.
Ferguson, C. E., 1969. The Neoclassical Theory of Production and Distribution. Cambridge University Press (Reprint 1979).
Felipe, J., Adams, F. G., 2005. "A Theory of Production" The Estimation of the Cobb-Douglas Function: A Retrospective View. Eastern Economic Journal 31 (3), 427-445.
Frondel, M., Schmidt, C. M., 2002. The capital-energy controversy: an artifact of cost shares? The Energy Journal 23(3), 53-79.
Grant, J. H., 1993. The translog approximate function : Substitution among inputs in manufacturing evaluated at sample means. Economics Letters 41 (3), 235-240.
Hicks, J., 1932. The Theory of Wages. (2nd, 1963 ed.) New-York..
Jorgenson, D. W., 1998. Growth, Volume 1: Econometric General Equilibrium Modeling. (1 ed.) (vols. 0262100738) The MIT Press.
Koetse, M. J., de Groot, H. L. F., Florax, R. J. G. M., 2008. Capital-energy substitution and shifts in factor demand: A meta-analysis, Energy Economics 30, 2236-2251.
Kümmel, R., Henn, J., Dietmar, L., 2002. Structural Change and Economic Dynamics 13, 415-433.
Kümmel, R., Strassl, W., Gossner, A., Eichhorn, W., 1985. Technical Progress and Energy Dependent Production Functions, Journal of Economics 3, 285-311.
Kmenta, J. 1967. On Estimation of the CES Production Function. International Economic Review 8 (180–89).
Landa Rivera, G., Reynès, F., Islas Cortes, I., Bellocq, F.-X., & Grazi, F., 2016. Towards a low carbon growth in Mexico: Is a double dividend possible? A dynamic general equilibrium assessment. Energy Policy, 96, 314-317.
Lemoine, M., Mazzi, G. L., Monperrus-Veroni, P., Reynès, F., 2010. A new production function estimate of the euro area output gap. Journal of Forecasting 29 (1-2), 29-53.
León-Ledesma, M. A., McAdam, P., Willman, A., 2010. Identifying the elasticity of substitution with biased technical change. American Economic Review, 100(4), 1330–1357.
McFadden, D., 1963. Constant Elasticity of Substitution Production Functions. The Review of Economic Studies 30 (2), 73-83.
Mishra, S. K., 2010. A Brief History of Production Functions. The IUP Journal of Managerial Economics 0 (4), 6-34.
Perroni, C., Rutherford, T. F., 1995. Regular flexibility of nested CES functions. European Economic Review 39 (2), 335-343.
Prywes, M., 1986. A nested CES approach to capital-energy substitution, Energy Economics 8, 22-28.
Robinson, J., 1933. The Economics of Imperfect Competition. New-York.
Roy, J., Sanstad A. H., Sathaya J. A., Khaddaria R, 2006. Substitution and Price Elasticity Estimates Using Inter-country Pooled Data in a Translog Cost Model. Energy Economics, 28(5-6), 706-719.
Samuelson, P. A., 1979. Paul Douglas's Measurement of Production Functions and Marginal Productivities. The Journal of Political Economy 87 (5), 923-939.
Sato, K., 1967. A Two-Level Constant-Elasticity-of-Substitution Production Function. The Review of Economic Studies 34 (2), 201-218.

Thompson, H., 2006. The applied theory of energy substitution in production, Energy Economics 28, 410-425.
Uzawa, H., 1962. Production Functions with Constant Elasticities of Substitution. The Review of Economic Studies 29 (4), 291-299.
Van der Werf, E., 2008. Production functions for climate policy modeling: An empirical analysis, Energy Economics 30, 2964-2979.
Varian, H., 1992. Microeconomic Analysis, W. W. Norton & Company, Inc., Third Edition.

Appendix A: Demonstration of Equation (16)

[Online supplement]

Combining the first order conditions (14) to the definition of the ES (9) gives:

$$\dot{X}_i - \dot{X}_j = -\eta_{ij}(\dot{P}_i^X - \dot{P}_j^X) \quad \text{(A.1)}$$

Inserting (A.1) into the production function (2) solving for $\dot{X}_i$ gives:

$$\dot{Q} = \varphi_i \dot{X}_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} \varphi_j \left(\dot{X}_i + \eta_{ij}(\dot{P}_i^X - \dot{P}_j^X) \right) \quad \text{(A.2)}$$

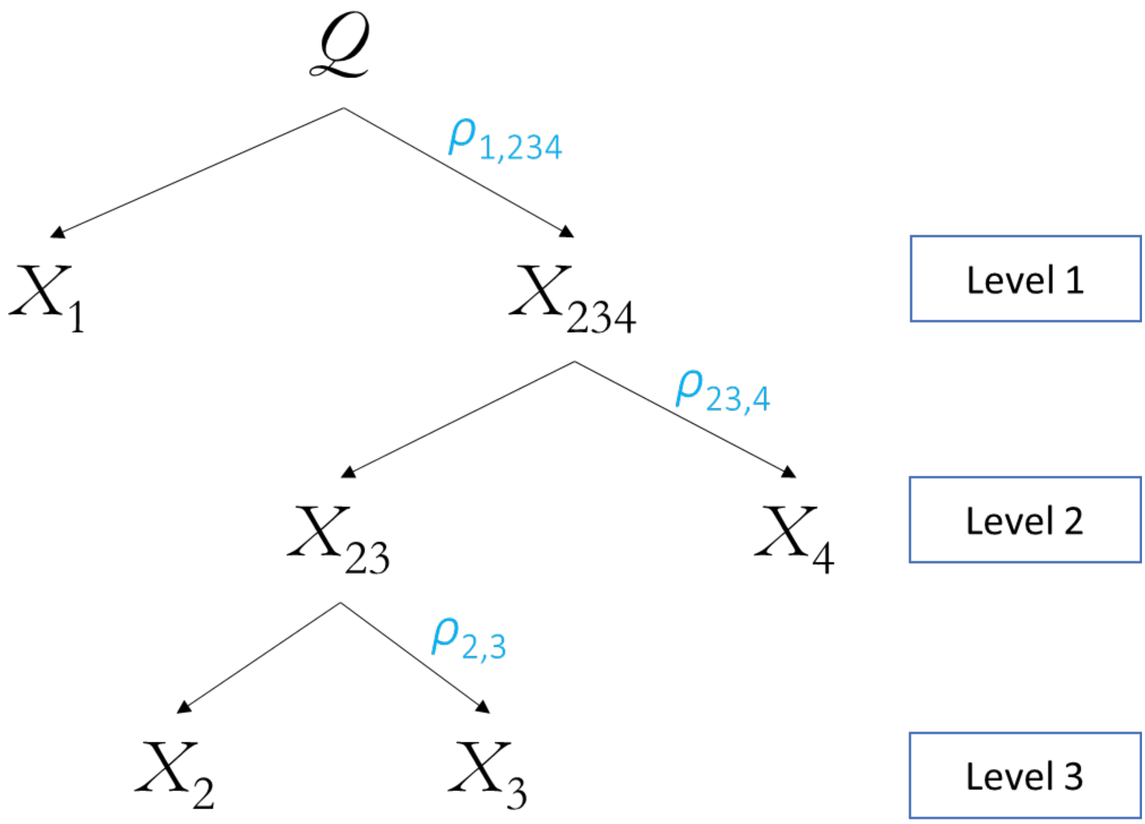
Rearranging (A.2) gives (16).

Appendix B: Example of nested CES functions with 4 inputs

[Online supplement]

In general equilibrium modeling and econometric studies, it is common to have an additional input: material (that is non energy intermediary consumption). This leads to a production technology with 4 inputs: capital (X_2), labor (X_4), energy (X_3) and material (X_1): also known as KLEM. Figure 2 shows a commonly used nested structure with 4 inputs. At the first level, material (X_1) can be substituted with the aggregate capital/energy/labor (X_{234}) with an ES of $\rho_{1,234}$. At the second level, the aggregate capital/energy (X_{23}) is a substitute to labor (X_4) with an ES of $\rho_{23,4}$. At the third level, capital (X_2) can be substituted to energy (X_3) with an ES of $\rho_{2,3}$.

Figure 2. Example of nested CES function with 4 inputs



The demand for input (16) can be used to represent this nested structure, replacing eventually Q and P_i^X by the relevant aggregate. This leads to the following linear system of Equations:

$$\dot{X}_1 = \dot{Q} - \rho_{1,234}\varphi_{234/1234}(\dot{P}_1^X - \dot{P}_{234}^X) \quad \text{(B.1)}$$

$$\dot{X}_2 = \dot{X}_{23} - \rho_{2,3}\varphi_{3/23}(\dot{P}_2^X - \dot{P}_3^X) \quad \text{(B.2)}$$

$$\dot{X}_3 = \dot{X}_{23} - \rho_{2,3}\varphi_{2/23}(\dot{P}_3^X - \dot{P}_2^X) \quad \text{(B.3)}$$

$$\dot{X}_4 = \dot{X}_{234} - \rho_{23,4}\varphi_{23/234}(\dot{P}_4^X - \dot{P}_{23}^X) \quad \text{(B.4)}$$

$$\dot{X}_{23} = \dot{X}_{234} - \rho_{23,4}\varphi_{4/234}(\dot{P}^X_{23} - \dot{P}^X_4) \quad \text{(B.5)}$$

$$\dot{X}_{234} = \dot{Q} - \rho_{1,234}\varphi_{1/1234}(\dot{P}^X_{234} - \dot{P}^X_1) \quad \text{(B.6)}$$

Where $\varphi_{234/1234} = (1-\varphi_1)$ is the share of the aggregated X_{234} into the production and $\varphi_{3/23} = \varphi_3/(1-\varphi_1-\varphi_4)$ is the share of the input X_3 into the aggregated X_{23}. Following the same logic, $\varphi_{2/23} = \varphi_2/(1-\varphi_1-\varphi_4)$, $\varphi_{23/234} = (\varphi_2+\varphi_3)/(1-\varphi_1)$, $\varphi_{4/234} = \varphi_4/(1-\varphi_1)$, $\varphi_{1/1234} = \varphi_1$. The price aggregates are: $\dot{P}^X_{234} = (\varphi_2\dot{P}^X_2 + \varphi_3\dot{P}^X_3 + \varphi_4\dot{P}^X_4)/(1-\varphi_1)$ and $\dot{P}^X_{23} = (\varphi_2\dot{P}^X_2 + \varphi_3\dot{P}^X_3)/(1-\varphi_1-\varphi_4)$.

Solving the above system in order to eliminate the (price) aggregates, it is straightforward to derive the explicit production factors demand as defined in (16) with n = 4. As shown in de proof below, after some algebra, we find the following ES between each pair of inputs:

$$\begin{aligned} &\eta_{1,2} = \eta_{1,3} = \eta_{1,4} = \rho_{1,234} \\ &\eta_{2,3} = \frac{\rho_{2,3}}{1-\varphi_1-\varphi_4} - \frac{\rho_{1,234}\cdot\varphi_1}{1-\varphi_1} - \frac{\rho_{23,4}\cdot\varphi_4}{(1-\varphi_1)(1-\varphi_1-\varphi_4)} \\ &\eta_{2,4} = \eta_{3,4} = \frac{\rho_{23,4} - \rho_{1,234}\cdot\varphi_1}{1-\varphi_1} \end{aligned} \quad \text{(B.7)}$$

As in the case of a nested structure with 3 inputs, the sign of the effective ES between inputs intervening at the lower level of the nest is ambiguous. From Equation (B.7), it is easy to show that all η_{ij} are positive if:

$$\begin{aligned} &\eta_{1,2} = \eta_{1,3} = \eta_{1,4} > 0 \text{ if } \rho_{1,234} > 0 \\ &\eta_{2,3} > 0 \text{ if } \rho_{2,3} > \rho_{1,234}\cdot\varphi_1(1-\phi) + \rho_{23,4}\phi \quad \text{with } \phi = \frac{\varphi_4}{1-\varphi_1} \in [0;1] \\ &\eta_{2,4} = \eta_{3,4} > 0 \text{ if } \rho_{23,4} > \rho_{1,234}\cdot\varphi_1 \end{aligned} \quad \text{(B.8)}$$

Similarly to the case with 3 inputs, these conditions are always satisfied (whatever the values of the shares φ_i if the ES defined in the lower levels of the nest are higher than the ones at the higher levels, that is if $\rho_{2,3} > \rho_{23,4} > \rho_{1,234}$.

Proof: Derivation of Equation (B.7)

Equation (B.1) can be reformulated as follows:

$$\dot{X}_1 = \dot{Q} - \rho_{1,234}(1-\varphi_1)\frac{\varphi_2(\dot{P}_1^X - \dot{P}_2^X) + \varphi_3(\dot{P}_1^X - \dot{P}_3^X) + \varphi_4(\dot{P}_1^X - \dot{P}_4^X)}{(1-\varphi_1)} \quad \text{(B.9)}$$

Equation (16) for $i = 1$ and $n = 4$ is:

$$\dot{X}_1 = \dot{Q} - \eta_{12}\varphi_2(\dot{P}_1^X - \dot{P}_2^X) - \eta_{13}\varphi_3(\dot{P}_1^X - \dot{P}_3^X) - \eta_{14}\varphi_4(\dot{P}_1^X - \dot{P}_4^X) \quad \text{(B.10)}$$

(B.9) is equivalent to (B.10) if $\eta_{1,2} = \eta_{1,3} = \eta_{1,4} = \rho_{1,234}$ as stated in Equation (B.7).

Inserting Equation (B.6) into (B.5) and then into (B.2) gives:

$$\dot{X}_2 = \dot{Q} - \rho_{1,234}\varphi_{1/1234}(\dot{P}_{234}^X - \dot{P}_1^X) - \rho_{23,4}\varphi_{4/234}(\dot{P}_{23}^X - \dot{P}_4^X) - \rho_{2,3}\varphi_{3/23}(\dot{P}_2^X - \dot{P}_3^X) \quad \text{(B.11)}$$

Adding $\dot{P}_2^X - \dot{P}_2^X$ in the terms $(\dot{P}_{234}^X - \dot{P}_1^X)$ and $(\dot{P}_{23}^X - \dot{P}_4^X)$ and rearranging, (B.11) can be reformulated as:

$$\begin{aligned}\dot{X}_2 = \dot{Q} &- (\dot{P}_2^X - \dot{P}_1^X)\rho_{1,234}\varphi_1 \\ &- (\dot{P}_2^X - \dot{P}_3^X)\left[\rho_{2,3}\varphi_{3/23} - \rho_{1,234}\varphi_1\varphi_{3/234} - \rho_{23,4}\varphi_{4/234}\varphi_{3/23}\right] \\ &- (\dot{P}_2^X - \dot{P}_4^X)\left[\rho_{23,4}\varphi_{4/234} - \rho_{1,234}\varphi_1\varphi_{4/234}\right]\end{aligned} \quad \text{(B.12)}$$

Which is equivalent to:

$$\begin{aligned}\dot{X}_2 = \dot{Q} &- (\dot{P}_2^X - \dot{P}_1^X)\varphi_1\left[\rho_{1,234}\right] \\ &- (\dot{P}_2^X - \dot{P}_3^X)\varphi_3\left[\frac{\rho_{2,3}}{1-\varphi_1-\varphi_4} - \frac{\rho_{1,234}\varphi_1}{1-\varphi_1} - \frac{\rho_{23,4}\varphi_4}{(1-\varphi_1)(1-\varphi_1-\varphi_4)}\right] \\ &- (\dot{P}_2^X - \dot{P}_4^X)\varphi_4\left[\frac{\rho_{23,4} - \rho_{1,234}\varphi_1}{1-\varphi_1}\right]\end{aligned} \quad \text{(B.13)}$$

The terms between […] can be identified to the ES of Equation (16) for *i* = 2 and *n* = 4. As stated in Equation (B.7), we get $\eta_{2,3} = \frac{\rho_{2,3}}{1-\varphi_1-\varphi_4} - \frac{\rho_{1,234}\cdot\varphi_1}{1-\varphi_1} - \frac{\rho_{23,4}\cdot\varphi_4}{(1-\varphi_1)(1-\varphi_1-\varphi_4)}$ and $\eta_{2,4} = \frac{\rho_{23,4}-\rho_{1,234}\cdot\varphi_1}{1-\varphi_1}$.

Inserting Equation (B.6) into (B.5) and then into (B.3) gives:

$$\dot{X}_3 = \dot{Q} - \rho_{1,234}\varphi_{1/1234}(\dot{P}_{234}^X - \dot{P}_1^X) - \rho_{23,4}\varphi_{4/234}(\dot{P}_{23}^X - \dot{P}_4^X) - \rho_{2,3}\varphi_{2/23}(\dot{P}_3^X - \dot{P}_2^X) \qquad \text{(B.14)}$$

Rearranging in the same way as we did for $\dot{X}_2$ gives:

$$\begin{aligned}\dot{X}_3 = \dot{Q} &- (\dot{P}_3^X - \dot{P}_1^X)\varphi_1\left[\rho_{1,234}\right] \\ &- (\dot{P}_3^X - \dot{P}_2^X)\varphi_2\left[\frac{\rho_{2,3}}{1-\varphi_1-\varphi_4} - \frac{\rho_{1,234}\varphi_1}{1-\varphi_1} - \frac{\rho_{23,4}\varphi_4}{(1-\varphi_1)(1-\varphi_1-\varphi_4)}\right] \\ &- (\dot{P}_3^X - \dot{P}_4^X)\varphi_4\left[\frac{\rho_{23,4}-\rho_{1,234}\varphi_1}{1-\varphi_1}\right]\end{aligned} \qquad \text{(B.15)}$$

The terms between […] corresponds to the result stated in Equation (B.7).

Inserting Equation (B.6) into (B.4) and using the same approach, we get the relation for $\dot{X}_4$ which corresponds to the outcome stated in Equation (B.7):

$$\begin{aligned}\dot{X}_4 = \dot{Q} &- (\dot{P}_4^X - \dot{P}_1^X)\varphi_1\left[\rho_{1,234}\right] \\ &- (\dot{P}_4^X - \dot{P}_2^X)\varphi_2\left[\frac{\rho_{23,4}-\rho_{1,234}\varphi_1}{1-\varphi_1}\right] \\ &- (\dot{P}_4^X - \dot{P}_3^X)\varphi_3\left[\frac{\rho_{23,4}-\rho_{1,234}\varphi_1}{1-\varphi_1}\right]\end{aligned} \qquad \text{(B.16)}$$